Degenerations of cohomology rings

Joint work with Bill Graham

Reference: E-Graham (w/appendix with Richmond): IMRN 2013, or arXiv 1104:1415

plus work in progress (stasis?)

Motivation

[BK] Belkale-Kumar

"Eigenvalue problem and a new cup product in cohomology of flag varieties" Inventiones Math 2006

Y = G/P, complex flag variety, $m = \dim(H^2(Y))$.

BK gave family of ring structures on $H^*(Y)$ parametrized by $t \in \mathbb{C}^m$.

We interpret this family using Lie algebra cohomology
and extend it to some non-Kahler homogeneous spaces

Notation

$\mathfrak{g}$ complex semisimple Lie algebra

G group of inner automorphisms of $\mathfrak{g}$

$\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$, Cartan subalgebra and Borel subalgebra of $\mathfrak{g}$

$T \subset B$ corresponding subgroups of G

$X = G/B = \cup_{w \in W} X_w$ Schubert cell decomposition

$W = N_G(T)/T$ Weyl group

$H^*(X) = \sum_{w \in W} \mathbb{C} S_w$

$\int_{X_y} S_w = \delta_{y,w}$

Kostant, Kumar gave explicit differential form representatives for S_w

Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. $\mathfrak{n}_-$ opposite nilradical of $\mathfrak{g}$

Explicit isomorphism
$\chi : H_*(\mathfrak{n} \oplus \mathfrak{n}_-)^{\mathfrak{t}} \cong H^*(\mathfrak{g}, \mathfrak{t}) = H^*(G/B)$

For each $w \in W$, there is easily written basis element $e_w \in H_*(\mathfrak{n} \oplus \mathfrak{n}_-)^{\mathfrak{t}}$

Indeed, if ∂ is the degree -1 operator computing Lie algebra homology, the Laplacian $\partial\partial^* + \partial^*\partial$ can be diagonalized with respect to basis given by wedges of root vectors, and its kernel is isomorphic to the homology of ∂

Then $S_w = \chi(e_w)$

Explain isomorphism χ

Consider $\mathfrak{g} \oplus \mathfrak{g}$ and its diagonal subalgebra $\mathfrak{g}_\Delta$

Regard $\mathfrak{g}_\Delta \in \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$, where $n = \dim(\mathfrak{g})$

$A := \{(t, t^{-1}) : t \in T\} \subset G \times G$

Let $\mathfrak{r} = (0, \mathfrak{n}_-) \oplus (\mathfrak{n}, 0)$

Gr_0 equals subspaces $U \in \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ such that $U \cap \mathfrak{r} = 0$

$\phi : A \to \mathrm{Gr}_0$, $a \mapsto \mathrm{Ad}(a)(\mathfrak{g}_\Delta)$

The image of ϕ is $\mathbb{C}^{*l}$, $l = \dim(A)$.

The closure of the image is $\mathbb{C}^l$ with A-action along coordinate planes, with 2^l orbits.
Action contracts towards 0,
so each orbit meets any open neighborhood of 0

Write as $\phi : \mathbb{C}^l \to \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$, $\phi : z \to \mathfrak{g}_z$

$\mathfrak{g}_0 = \mathfrak{t}_\Delta + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-)$, $\mathfrak{t}_\Delta = \{(X, X) : X \in \mathfrak{t}\}$

Each $\mathfrak{g}_z \supset \mathfrak{t}_\Delta$.

$\mathfrak{r} \cong (\mathfrak{g}_s/\mathfrak{t}_\Delta)^*$ via Killing form

The complex $\wedge^\cdot(\mathfrak{g}_s/\mathfrak{t}_\Delta)^{*,\mathfrak{t}_\Delta}$ has differential computing relative Lie algebra cohomology $H^*(\mathfrak{g}_s, \mathfrak{t}_\Delta)$

Via $\mathfrak{r} \cong (\mathfrak{g}_s/\mathfrak{t}_\Delta)^*$, $\wedge^\cdot(\mathfrak{r})^{\mathfrak{t}_\Delta} \cong \wedge^\cdot(\mathfrak{g}_s/\mathfrak{t}_\Delta)^{*,\mathfrak{t}_\Delta}$
the complex $C^\cdot := \wedge^\cdot(\mathfrak{r})^{\mathfrak{t}_\Delta}$
acquires a differential d_z for each $z \in \mathbb{C}^l$.

The complex $C^\cdot$ has a degree -1 operator ∂ computing $H_*(\mathfrak{r})^{\mathfrak{t}_\Delta}$.

finite dimensional Hodge theory (Kostant)

Let $C^{\cdot}$ be a complex of finite dimensional vector spaces with

$d : C^k \to C^{k+1}$, $\partial : C^k \to C^{k-1}$.

fake Laplacian $L = d\partial + \partial d$

DEFINITION: d and ∂ are disjoint if
$\mathrm{Im}(d) \cap \ker(\partial) = \mathrm{Im}(\partial) \cap \ker(d) = 0$.

Remark: If $\partial = d^*$ with respect to some positive definite Hermitian metric on $C^{\cdot}$, then d and ∂ are disjoint, and L is really the Laplacian.

PROPOSITION: If d and ∂ are disjoint, then
(1) If $s \in \ker(S)$, then $ds = \partial s = 0$.
(2) The canonical map $\ker(L) \to H^*(C^{\cdot}, d)$,
$s \mapsto s + d(C^{\cdot})$ is an isomorphism.
(3) The canonical map $\ker(L) \to H_*(C^{\cdot}, \partial)$,
$s \mapsto s + \partial(C^{\cdot})$ is an isomorphism.

Hence, by composing isomorphisms to $\ker(L)$, we have an isomorphism $H_*(C^\cdot, \partial) \to H^*(C^\cdot, d)$, provided we know that d and ∂ are disjoint.

NEW ARGUMENT FOR DISJOINTNESS

In our situation, we have a family of degree 1 operators d_z and one degree -1 operator ∂.

LEMMA: $\dim \ker(d_z)$ and $\dim \mathrm{Im}(d_z)$ are independent of z.

Idea of proof: $\dim(H^*(C^\cdot, d_0)) = |W|$ by Kostant's theorem on $\mathfrak{n}$-homology. Since $\dim(H^*(C^\cdot, d_z)) = \dim(H^*(G/B)) = |W|$ for generic z, and rank of a family of linear operators cannot increase under specialization, lemma follows.

To prove disjointness:

(1) The condition for d_z and ∂ to be disjoint is an open condition on $z \in \mathbb{C}^l$, since condition on family in Grassmannian to have zero intersection with a fixed subspace is open (need Lemma)
(2) The condition for d_z and ∂ to be disjoint is constant on A-orbits
(3) $d_0 = \partial^*$, so d_0 and ∂ are disjoint.

Using (1) and (3), d_z and ∂ are disjoint in a neighborhood of 0
Using (2) and the fact that each A-orbit meets each neighborhood of 0, we see d_z and ∂ are disjoint for all s

CONCLUDE: By Hodge theory, for all $z \in \mathbb{C}^l$,
$H_*(C^\cdot, \partial) \cong H^*(C^\cdot, d_z)$

Further, the isomorphism can be made explicit.

Hence, for each $w \in W$, the generator
$e_w \in H_*(\mathfrak{n} \oplus \mathfrak{n}_-)^{\mathfrak{t}}$ gives $S_w \in H^*(\mathfrak{g}/\mathfrak{t}) = H^*(G/B)$

Note: This argument is inspired by an argument from E-Lu, Advances 1999.
In that paper, we also showed that the diffential forms S_w
are "Poisson harmonic" in an appropriate sense using the modular class.
We use this together with the Bruhat-Poisson structure
to show $\int_{X_y} S_w = \delta_{y,w}$.

Cup product and its deformation

Let R be the roots of $\mathfrak{t}$ in $\mathfrak{g}$, and
R^+ roots in $\mathfrak{b}$
$\{\alpha_1, \ldots, \alpha_l\}$ simple roots

$H^*(G/B) = \sum_{w \in W} \mathbb{C} S_w$

Cup product $S_u \cdot S_v = \sum_{w \in W} c_{uv}^w S_w$ for $u, v \in W$, $c_{uv}^w \in \mathbb{Z}$.

For $\alpha \in R^+$, write $\alpha = \sum_{i=1}^l k_i \alpha_i$.

Let $z_\alpha = \prod_{i=1}^l z_i^{k_i}$

For $w \in W$, let $F_w(z) := \prod_{\alpha \in R^+ \cap w^{-1} R^-} z_\alpha^2$.

Definition of Belkale-Kumar deformed cup product:

$s_u \odot s_v = \sum_{w \in W} \frac{F_w(z)}{F_u(z) F_v(z)} c_{uv}^w S_w$.

Notation: Let $H^*(G/B)_z$ be the space $H^*(G/B)$ with product $\odot$ specialized at $z \in \mathbb{C}^l$.

Some remarks:

(1) Belkale and Kumar proved the product $\odot$ is well-defined for all $z \in \mathbb{C}^l$, i.e., c_{uv}^w nonzero implies that the rational function $\frac{F_w(z)}{F_u(z)F_v(z)}$ is regular. Pechenik and Searles gave an alternate proof.

(2) Degeneration at $z = 0$ has the effect of degenerating some coefficients to 0. When $z = 0$, product seems to be significantly more computable. See Knutson-Purbhoo, Electron. J. Combin. 18 (2011) for nice combinatorial description of structure constants for the cohomology ring $H^*(G/B)_0$ for type A.

(3) One can do the same thing for $H^*(G/P)$, but I am omitting these cases from the talk to minimize notation. It is important to do this, since Ressayre proved that structure constants when $z = 0$ for all maximal parabolics gives irredundant conditions for geometric Horn problem, answering question of Belkale-Kumar.

Although Belkale-Kumar proof uses geometry, the family is defined formally. We wanted to better understand the family.

RECALL: Relative Lie algebra cohomology has ring structure from wedge product

$H^*(C^\cdot, d_z) \cong H^*(\mathfrak{g}_z, \mathfrak{t}_\Delta)$ is a ring.

Since $H_*(C^\cdot, \partial) \cong H^*(C^\cdot, d_z)$,
we have a family of ring structures on a vector space with
basis parametrized by $w \in W$.

Theorem: $H^*(\mathfrak{g}_z, \mathfrak{t}_\Delta) \cong H^*(G/B)_z$.

To prove this theorem, we have to identify the product from our basis with the Belkale-Kumar product. We do this by using the family to carry out the identification on $\mathbb{C}^{*l}$ and then use continuity.

Our approach: Should define Belkale-Kumar cup product using relative Lie algebra cohomology.

Generalization to real groups

Basic idea: Map $\mathbb{C}^l \to \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ is key feature of DeConcini-Procesi compactification of the group G, regarded as a symmetric space. Would like to generalize to other symmetric spaces. This works in a few cases.

$\mathfrak{g}_0$ real semisimple Lie algebra

G_0 group of inner automorphisms of $\mathfrak{g}_0$

$K_0 \subset G_0$ maximal compact subgroup

Iwasawa decomposition: $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{u}_0$

$\mathfrak{u}_{0,-}$ opposite nilradical

$\mathfrak{m}_0$ centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$

$\mathfrak{g}_0 = \mathfrak{u}_{0,-} + \mathfrak{m}_0 + \mathfrak{a}_0 + \mathfrak{u}_0$, direct sum decomposition

We can complexify everything in sight:

$\mathfrak{g} = \mathfrak{u}_- + \mathfrak{m} + \mathfrak{a} + \mathfrak{u}$

Assume $\mathfrak{g}_0$ is *nearly diagonal*, i.e., it has a unique G_0-conjugacy class of Cartan subalgebras. This happens in essentially 4 cases:

(1) $\mathfrak{g}_0$ is complex, so $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0$, $\mathfrak{k} = \mathfrak{g}_\triangle$, $\mathfrak{m} = \mathfrak{t}_\triangle$

(2) $\mathfrak{g}_0 = \mathfrak{su}^*(2n)$, so $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{C})$, $\mathfrak{k} = \mathfrak{sp}(2n, \mathbb{C})$, $\mathfrak{m} = \mathfrak{sp}(2, \mathbb{C})^n$

(3) $g_0 = \mathfrak{so}(2n-1, 1)$, so $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, $\mathfrak{k} = \mathfrak{so}(2n-1, \mathbb{C})$, $\mathfrak{m} = \mathfrak{so}(2n-2, \mathbb{C})$

(4) exceptional case, $\mathfrak{g} = E_6$, $\mathfrak{k} = F_4$, $\mathfrak{m} = \mathfrak{so}(8, \mathbb{C})$

The cases (2), (3), (4) correspond to connected Dynkin diagrams with an involution that does not interchange any two consecutive simple roots.

Remark: The *nearly diagonal* assumption gives exactly the symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ such that $\mathfrak{k}$ and $\mathfrak{m}$ have the same rank. Perhaps something is true beyond these cases.

However, our goal is to study $H^*(K/M) = H^*(\mathfrak{k}, \mathfrak{m})$, and if we don't assume equal rank, this is quite different from $H^*(G/P)$. We aren't yet brave enough to try without nearly diagonal assumption.

Consider Levi subalgebra $\mathfrak{l} = \mathfrak{m} + \mathfrak{a}$.

Let A, K, M be groups corresponding to $\mathfrak{a}, \mathfrak{k}, \mathfrak{m}$.

Let $n = \mathrm{dim}(\mathfrak{k})$, $\mathfrak{k} \in \mathrm{Gr}(n, \mathfrak{g})$

Let Gr_0 consist of subspaces $V \in \mathrm{Gr}(n, \mathfrak{g})$ such that $V \cap (\mathfrak{u}_- \oplus \mathfrak{a}) = 0$.

$\phi : A \to \mathrm{Gr}_0$, $\phi(a) = \mathrm{Ad}(a)(\mathfrak{k})$.

$\phi(A) \cong \mathbb{C}^{*l}$, and we can extend to a morphism $\phi : \mathbb{C}^l \to \mathrm{Gr}_0$, $\phi(z) = \mathfrak{k}_z$ (after DeConcini-Procesi)

Each $\mathfrak{k}_z \supset \mathfrak{m}$, and $\mathfrak{k}_0 = \mathfrak{m} + \mathfrak{u}$.

Idea: would like to show $H^*(\mathfrak{k}_z, \mathfrak{m})$ is independent of z

Can do this under the assumption that $\mathfrak{g}_0$ is nearly diagonal.

Further, the earlier disjointness argument works, giving an explicit isomorphism

$H_*(\mathfrak{u})^{\mathfrak{m}} \cong H^*(\mathfrak{k}, \mathfrak{m})$.

Kostant's work gives basis for $H_*(\mathfrak{u})^{\mathfrak{m}}$ parametrized by elements of W_K/W_M (Weyl group of K modulo Weyl group for M).

By applying the isomorphism to Kostant's classes, we obtain differential forms on K/M which give a basis of the cohomology.

One can write a formula for these differential forms. They are given by applying an explicit unipotent operator to Kostant's classes.

In case (2), $H^*(\mathfrak{k}, \mathfrak{m}) = H^*(Sp(2n)/Sp(2)^n)$ ($Sp(2n)$ and $Sp(2)^n$ are my notation for compact groups of type C_n and $(C_1)^n$).

$Sp(2n)/Sp(2)^n$ can be identified as quaternionic flag variety, and is non-Kahler. This means usual theory for complex generalized flag varieties does not apply.

These K/M have cell decompostions with even dimensional cells. We expect our differential forms to be dual to the cell basis of homology. For the case when $\mathfrak{g}_0$ is complex, $K/M = G/B$, and there is the Bruhat-Poisson structure, which makes the assertion relatively easy to verify. K/M is not a Poisson homogeneous space for the standard Poisson structure on K, so other methods are needed.

These methods give a Belkale-Kumar type family of cup products on $H^*(K/M)$ in the almost diagonal cases. We would like to connect this to a geometric Horn-type problem.

Thanks.